Differential embeddings into algebras of topological stable rank 1

Natalia Maślany

Introduction

Does Banach's open mapping theorem hold true for bilinear mappings?

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A natural example of a bilinear continuous surjection:

$$(f, g) \mapsto f \cdot g$$

Introduction

Fremlin's example, 2004

$$(\mathcal{C}_{\mathbb{R}}[0,1], \|\cdot\|_\infty)$$

Lack of openness of multiplication at (f, f), where

$$f(x) = x - \frac{1}{2} \quad (x \in [0,1]).$$

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Why?

Definition

A unital Banach algebra A has topological stable rank 1 ($\operatorname{tsr} A = 1$) when the set of all invertible elements in A is dense in A.

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Theorem (S.Draga, T.Kania, 2017)

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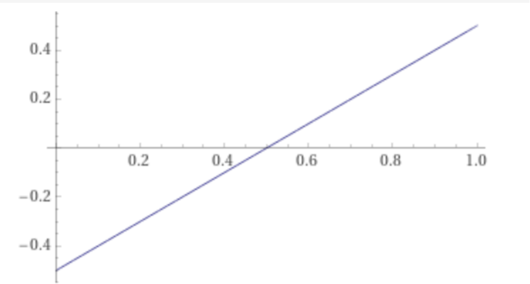

Examples

Examples of function algebras **with open multiplication**:

- spaces of complex continuous functions (on at most 1-dim spaces);
- spaces of bounded functions;
- spaces of functions of bounded p-variation variation $(1 \leq p < \infty)$.

The first two results: M. Balcerzak, E. Behrends, F. Botelho, A. Komisarski, A. Maliszewski, H. Renaud, F. Strobin, A. Wachowicz, W. Wilczyński, T. Kania, M.
More recent results: T. Canarias, A. Karlovich, E. Shargorodsky, S. Kowalczyk, M. Turowska.

Introduction

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Question

When is the algebra closed under inverses?

Introduction

Definition

When $i \colon A \to B$ is a unital continuous injective homomorphism of Banach algebras, we say that A admits norm-controlled inversion in B, if there exists $h \colon (0,\infty)^2 \to (0,\infty)$ so that for every $a \in A$, which is invertible in B, we have

$$\|a^{-1}\|_A \leqslant h(\|a\|_A, \|i(a^{-1})\|_B).$$

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Convention

We say that a **commutative (*-)semi-simple Banach (*-)algebra** admits norm-controlled inversion, if it has this property in $C(\Phi_A)$, when embedded by the Gelfand transform.

Introduction

inverse-closed $\not\Longrightarrow$ norm-controlled inversion (Nikolski, 1999)

The Wiener (convolution) algebra $\ell_1(\mathbb{Z})$ is a commutative Banach *-algebra without the norm-controlled inversion in $C(\mathbb{T})$.

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Definition

When $i\colon A \to B$ is a unital injective homomorphism of Banach algebras, then A is a differential subalgebra of B, if there is $D > 0$ such that for all $a, b \in A$ we have

$$\|ab\|_A \leqslant D(\|a\|_A \|i(b)\|_B + \|i(a)\|_B \|b\|_A).$$

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Lemma (K. Gröchenig, A. Klotz, 2013)

Differential *-subalgebras of C*-algebras have norm-controlled inversion.

Main theorem 1

Definition

Let A be a unital Banach *-algebra. For $a \in A$ we interpret $|a|^2$ as a^*a. We say that elements a, b in A are jointly non-degenerate, when $|a|^2 + |b|^2$ is invertible.

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Theorem 1 (Kania, M., 2023)

Suppose that A is a unital symmetric dual Banach *-algebra that is a dense differential subalgebra of $C(X)$. If A shares with X densely many points, then multiplication in A is open at pairs of elements that are jointly non-degenerate.

Main theorem 1

The idea of the proof of Th. 1.

Multiplication is open at (F, G) iff for any $\varepsilon > 0$ there is some $\delta > 0$ such that

$$B_A(F \cdot G, \delta) \subset B_A(F, \varepsilon) \cdot B_A(G, \varepsilon). \quad (1)$$

Condition (1) means that for any $H \in A$ with $\|H\| < \delta$ there are $f, g \in A$ such that

- $\|f - F\|_A < \varepsilon$
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How to do it?

- $F_0 := F$
- $G_0 := G$
- $H_0 := H$
- $F_{n+1} := F_n + \frac{H_n \overline{G_n}}{|F_n|^2 + |G_n|^2}$
- $G_{n+1} := G_n + \frac{H_n \overline{F_n}}{|F_n|^2 + |G_n|^2}$
- $H_{n+1} := -\frac{H_n^2 \overline{F_n G_n}}{(|F_n|^2 + |G_n|^2)^2}$

Then for all $n \in \mathbb{N}$

$$F_n G_n + H_n = FG + H$$

Main theorem 1

Example 1

Let A be a (complex) reflexive Banach space with a K-unconditional basis $(e_\gamma)_{\gamma\in\Gamma}$ ($K \geqslant 1$). Then A is naturally a Banach *- algebra when endowed with multiplication

$$a \cdot b = \sum_{\gamma\in\Gamma} a_\gamma b_\gamma e_\gamma \quad \left(a = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma, \;\; b = \sum_{\gamma\in\Gamma} b_\gamma e_\gamma \in A \right)$$

and coordinate-wise complex conjugation. Let $A^\#$ denote the unitisation of A. Then $A^\#$ has open multiplication.

Main theorem 1

Proof

Since the basis $(e_\gamma)_{\gamma\in\Gamma}$ is K-unconditional, we have

$$\begin{aligned}\|ab\|_A = \Big\|\sum_{\gamma\in\Gamma} a_\gamma b_\gamma e_\gamma\Big\|_A &\leqslant K\Big\|\sum_{\gamma\in\Gamma} a_\gamma \cdot \|b\|_{\ell_\infty(\Gamma)} \cdot e_\gamma\Big\|_A \\ &= K\|a\|_A\|b\|_{\ell_\infty(\Gamma)} \leqslant K(\|a\|_A\|b\|_{\ell_\infty(\Gamma)} + \|a\|_{\ell_\infty(\Gamma)}\|b\|_A).\end{aligned}$$

This means that $A^\#$ is a differential subalgebra of $c(\Gamma)$, the unitisation of the algebra of functions that vanish at infinity on Γ. Since the formal inclusion from $A^\#$ to $c(\Gamma)$ has dense range, the conclusion follows.

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Proof

Since the basis $(e_\gamma)_{\gamma\in\Gamma}$ is K-unconditional, we have

$$\|ab\|_A = \left\| \sum_{\gamma\in\Gamma} a_\gamma b_\gamma e_\gamma \right\|_A \leqslant K \left\| \sum_{\gamma\in\Gamma} a_\gamma \cdot \|b\|_{\ell_\infty(\Gamma)} \cdot e_\gamma \right\|_A$$
$$= K\|a\|_A\|b\|_{\ell_\infty(\Gamma)} \leqslant K(\|a\|_A\|b\|_{\ell_\infty(\Gamma)} + \|a\|_{\ell_\infty(\Gamma)}\|b\|_A).$$

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Question

What are further examples of (dual) Banach algebras that are approximable by invertible elements? What about algebras of Lipschitz functions on zero-dimensional compact spaces?

Main theorem 2

Definition

A map $f\colon X \to Y$ is uniformly open if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x \in X \quad B(f(x), \delta) \subseteq f[B(x, \varepsilon)]$$

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- not weakly open, when $\dim X > 1$.

Main theorem 2

Theorem 2 (T. Kania, M., 2023)

Let X be a compact space. Then the following conditions are equivalent for the algebra $C(X)$ of continuous complex-valued functions on X:

- $C(X)$ has open multiplication,
- $C(X)$ has uniformly open multiplication,
- the covering dimension of X is at most 1.

Moreover, the algebras $C(X)$ have equi-uniformly open multiplications for all compact spaces of dimension at most 1.

Main theorem 2

The idea of the proof of Th. 2.

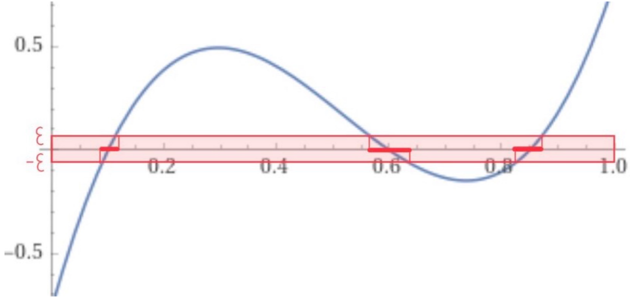
The proof is split into **3 cases**:

- a reduction to spaces being topological (planar) realisations of **graphs** (adapting unpublished result of Behrends)
- the result for all compact **metric** spaces of dim $\leqslant 1$
- the general one-dim **non-metrisable** case (dim $\leqslant 1$)

Main theorem 2

How to do it?

$$(|F|^2 + |G|^2)(x),\ x \in [0,1]$$



Main theorem 2

What to do if $|F|^2 + |G|^2 < \varepsilon$?

We have to find $f, g \in C([a, b])$ such that

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Define

$$\Psi(x) := (FG + H)(x) \quad \text{for} \quad x \in [a, b].$$

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Note that $f(a), f(b)$ are already known.
Define f in such a way that

$$|f(x)| \geq \sqrt{|\Psi(x)|} \quad \text{for all} \quad x \in [a, b]$$

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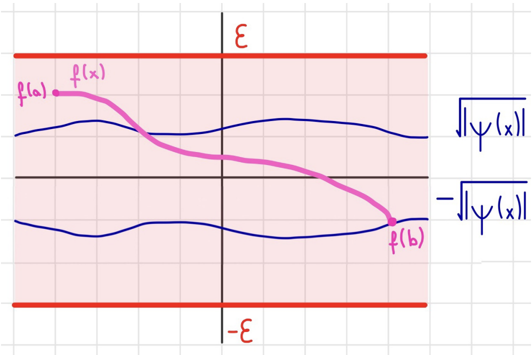
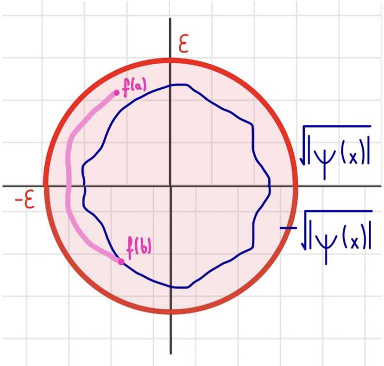
and function g as

$$g(x) := \begin{cases} \frac{\Psi(x)}{f(x)} & \text{if } f(x) \neq 0 \\ 0 & \text{if } f(x) = 0. \end{cases}$$

Main theorem 2



(a) impossible in the real case



(b) possible in the complex case

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Case 1: X is a topological realisation of a graph

We consider a partition of X into finitely many intervals, $\bigcup_{j=1}^{k}[a_j, b_j]$ and for each interval we apply an analogous procedure as previously.

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We consider a partition of X into finitely many intervals, $\bigcup_{j=1}^{k}[a_j, b_j]$ and for each interval we apply an analogous procedure as previously.

Case 2: X is a compact **metric space** of dim $\leqslant 1$

Every compact metric space of dim $\leqslant 1$ is an inverse limit of planar graphs (Freudenthal, 1937).
We apply the theorem (S. Draga, T. Kania, 2018) which states that if a net of Banach algebras has equi-uniformly open multiplication, the same holds for its direct limit.

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Case 3: X is an arbitrary compact space of dim $\leqslant 1$

Every compact space of dim $\leqslant n$ is an inverse limit of compact metric spaces of dim $\leqslant n$ (S. Mardešić, 1960).

Main theorem 3

Question (Balcerzak, Behrends, Strobin, 2016)

Is the Cauchy product in $\ell_1(\mathbb{N}_0)$ open or even uniformly open?

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Note: Cauchy product is nothing but convolution w.r.t. the semigroup $(\mathbb{N}_0, +)$.

Semigroup algebras

Let $(S, \cdot)$ be a semigroup. Convolution in $\ell_1(S)$ is defined as

$$(x_s)_{s\in S} * (y_s)_{s\in S} = \sum_{s\in S} \left(\sum_{s=r\cdot t} x_r y_t \right) \mathrm{e}_s \quad \left((x_s)_{s\in S}, (y_s)_{s\in S} \in \ell_1(S)\right),$$

where $(\mathrm{e}_s)_{s\in S}$ is the unit vector basis of $\ell_1(S)$.

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Fact

The boundary of the set of invertible elements within a unital Banach Algebra consists of *topological zero divisors*, *i.e.*, elements a that satisfy

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Corollary

In $\ell_1(\mathbb{N}_0)$, element e_1 is not a topological zero divisor, so convolution in $\ell_1(\mathbb{N}_0)$ is **not open**.

Main theorem 3

Theorem (Draga, Kania, 2018)

Convolution in $\ell_1(\mathbb{Z})$ is not uniformly open.

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Theorem 3 (T. Kania, M., 2023)

Let G be an Abelian group of unbounded exponent, i.e., $\sup_{g\in G} \mathrm{ord}(g) = \infty$. Then convolution in $\ell_1(G)$ is not uniformly open.

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Question 1

Suppose that G is an abelian group whose elements have uniformly bounded ranks. Does $\ell_1(G)$ have open convolution?

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Question 1

Suppose that G is an abelian group whose elements have uniformly bounded ranks. Does $\ell_1(G)$ have open convolution?

Question 2

For which semigroups S is convolution in $\ell_1(S)$ (uniformly) open? Particular examples welcome.

Thank you for your attention!

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